Two-dimensional reattaching jet flows including the effects of curvature on entrainment

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The analysis given previously for predicting the average pressure and length of the region of recirculating flow enclosed by a low-speed turbulent jet, issuing parallel to a flat plate, has been modified to take into account the different rates of entrainment by the two edges of the curved jet, the initial mixing region and the pressure forces near reattachment. There is improved correlation between theory and experiment. The analysis has been applied to the flow due to a jet emerging at an angle to a flat plate, and gives good prediction of the length and average pressure of the recirculation region for a particular value of an entrainment-ratio parameter.

Curvature has a considerable effect on the rates of entrainment, but a first-order mixing-length theory indicates that this need not necessarily be accompanied by a marked deviation in jet velocity profile from that of a plane jet.

1. Introduction

In experiments previously reported (Sawyer 1960) on the flow due to a two-dimensional turbulent jet of incompressible fluid issuing parallel to a flat plate, it was found that the measured velocity profiles of the jet as it curved towards the plate exhibited no obvious asymmetry. This was in apparent disagreement with expectations based on the arguments first put forward by Prandtl (1929), which indicate that there should be an enhanced mixing in the outer portion of a curved jet and a reduced mixing in the inner portion due to the influence of centrifugal forces on the parcels of fluid which transfer momentum from layer to layer. The experiments of Wilcken (1930), Wattendorf (1935) and others on the effects of curvature on turbulent boundary layers and fully developed channel flow, as well as Newman's (1961) measurements of wall jets blowing round circular cylinders, showed that curvature has a considerable effect on the development of turbulent shear layers. In each of these configurations the effect was in agreement with Prandtl's momentum-transfer arguments.

The author has measured the growth of two-dimensional turbulent wall jets blowing round surfaces constructed to maintain the ratio of jet thickness b to surface radius of curvature R at a constant value along the length of the jet (Sawyer 1962). For b/R of the order of 0·05, the rate of spread of the curved jet was found to be increased or decreased by approximately 50 % as compared with that of a plane wall jet, for a convex or concave surface, respectively.

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In view of these considerations, it was somewhat surprising to discover that the velocity profiles of the curved jet which was free to entrain fluid from both sides (i.e. a jet across which there was a pressure difference) were nearly symmetrical, even though the corresponding value of b/R was of the order of 0·06 for both sides of the jet. It was also found that the total rate of spread of the jet was almost identical to that of a plane jet. It appeared, therefore, that there must be a flow of fluid across the jet centre-line for the observed symmetry of the velocity profiles to be compatible with quite different entrainment rates at the two sides of the jet.

For the case of a two-dimensional jet issuing parallel to a flat plate, a simple model of the flow was proposed (Sawyer 1960) following work by Dodds (1960), in which the mean velocity of the jet was assumed to be given by $u/U = \operatorname{sech}^2 \eta$, where $\eta = \sigma Y/X$ and where X, Y are co-ordinates measuring distances along and perpendicular to the jet centre-line. This is the solution obtained by Görtler (1942) on the basis of constant eddy viscosity across the jet. Reichardt (1942) found that the empirical constant σ has the value 7·67 for a plane jet. The analysis did not take into account explicitly the different entrainment rates at the two edges of the curved jet.

In this paper, the previous analysis is reframed to include the effects of different rates of entrainment along the inner and outer edges of the jet, although it is still assumed that sufficiently far from the slot the velocity profile is approximately that of a free jet, i.e. may be adequately represented by $u/U = \operatorname{sech}^2 \eta$. The effect of non-uniformity of the jet velocity profile at the slot is also introduced, and the equation expressing the conservation of momentum parallel to the plate at the reattachment point of the jet is modified to take into account the pressure differences in the neighbourhood of reattachment. This gives a more realistic model of the flow, and this model is used to predict the length and average pressure of the recirculation region associated with a jet emerging at an angle to a flat plate. These predictions are compared with Bourque's (1959) measurements.

It is possible to formulate Prandtl's momentum transfer arguments, which are briefly mentioned at the beginning of this section, in terms of simple mixing-length theory, and to obtain from this an expression for the mixing length which includes a first-order term in $1/R$, where R is the local radius of curvature of the streamlines (Sawyer 1962). In § 5 this is used to calculate the velocity profiles of a curved jet.

2. The modified analysis

The notation used is shown in figure 1 and definitions of symbols appear in the Appendix.

The velocity profile of a plane turbulent jet is represented quite closely by $u/U = \operatorname{sech}^2 \eta$, where $\eta = \sigma Y/X$, except at the edges of the jet where there is considerable intermittency of the flow. For many purposes it is physically more meaningful to use an entrainment parameter E rather than the jet-spread parameter σ. Following Head (1958), if E is defined to be the rate of increase in volume flow per unit span of the turbulent jet with downstream distance, suitably non-dimensionalized, then

$$E = \frac{1}{U}\frac{d}{dX}\int_{-\infty}^{+\infty} u\,dY = \frac{1}{\sigma},$$

if $u/U = \mathrm{sech}^2(\sigma Y/X)$ and $U \propto X^{-\frac{1}{2}}$, which follows from the assumption of similar velocity profiles, since the jet momentum per unit span, J, is independent of X. Thus $E = 0{\cdot}130$ for a plane jet. If E_1 and E_2 are the non-dimensional rates of entrainment by the inner and outer edges of a curved jet, where the reference velocity is the maximum profile velocity U as before, then $E_1 + E_2 = E$ taking the velocity profile $u/U = \mathrm{sech}^2\eta$ for the curved jet, and $1 - 2E_1/E$ gives a measure of the difference in the entrainment rates at the two sides of the jet. In a first-order theory such as that set out in § 5, it is seen that $(1 - 2E_1/E) \propto b/R$.

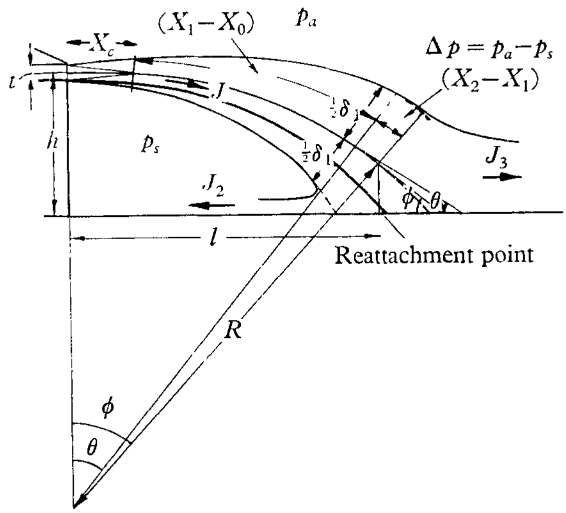

FIGURE 1. Notation used in the analysis of the flow of a jet from the top of a step.

It is now necessary to calculate the region of initial growth of the jet from the slot to the point downstream of which the velocity profile is adequately described by $u/U = \mathrm{sech}^2\eta$, where $\eta = Y/EX$. Over this initial region the effects of jet curvature are small and are neglected in the calculation which proceeds as follows.

Consider first the development of a plane jet of initially uniform velocity profile. This takes place in three distinct regions (Townsend 1956, p. 172), of which the central region is a transition between the first, where the jet consists of a potential core bounded by two turbulent mixing layers, and the third, where the jet flow is self-preserving. For the purposes of the calculation the transition region is ignored, so that the jet is taken to grow as a free jet with velocity profile $u/U = \mathrm{sech}^2\eta$ from the point where the initial mixing layers have grown sufficiently to have entrained the whole of the potential core (figure 2).

The mixing layers bounding the potential core are of the type investigated by Liepmann & Laufer (1947), who showed that the velocity distribution within such a layer is expressed as $u/U = f(\xi)$, where $\xi = 12{\cdot}0\,Y/X$ and where $f(\xi)$ is approximated by the error integral $\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\xi} e^{-\xi^2}\,d\xi$. In figure 2 it is shown that the mixing-layer velocity distribution $u/U = f(\xi)$ measured by Liepmann & Laufer corresponds quite closely to $u/U = \operatorname{sech}^2\eta$, if the scales of ξ and η are chosen so that $\xi = 1{\cdot}19, -0{\cdot}135$ correspond to $\eta = 0, 1$. Thus, if the mixing layers bounding

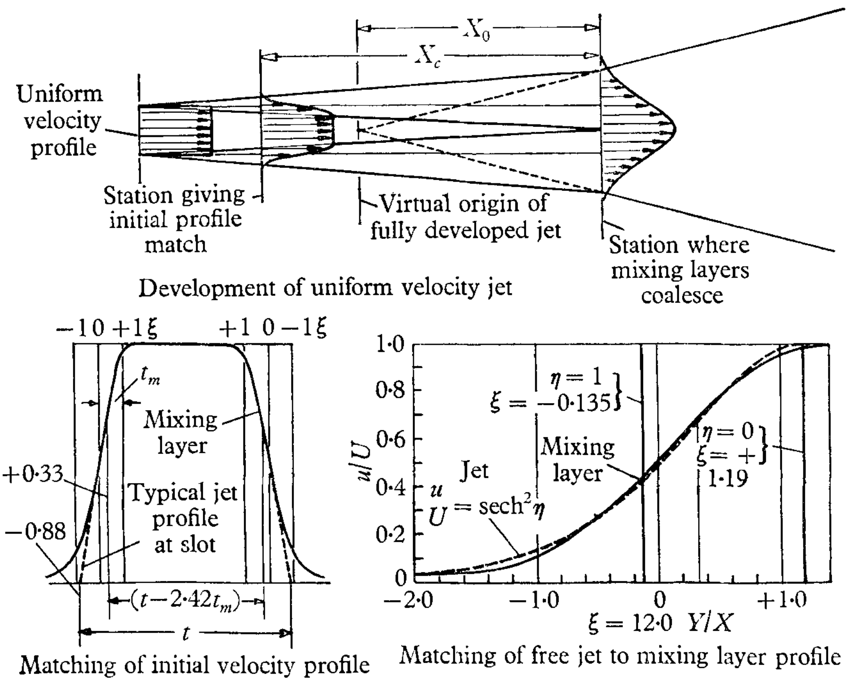

FIGURE 2. Initial development of a two-dimensional jet.

the potential core are assumed to grow at a rate given by $u/U = f(\xi)$ and $\xi = 12{\cdot}0\,Y/X$ up to the point where the lines $\xi = 1{\cdot}19$ intersect on the centre-line of the jet, then the total velocity distribution of the two mixing layers at this point is a good approximation to $u/U = \operatorname{sech}^2\eta$. From this point, therefore, it is reasonable to assume that the jet continues to grow as a free jet. The free jet will have a virtual origin at a distance X_0, say, upstream of this point, where the mixing layers coalesce.

Now the lines $\xi = 0{\cdot}33$ are parallel to the potential core (Liepmann & Laufer), and so the distance between these lines is the initial thickness of the uniform velocity jet. Thus the distance between the points $\eta = 0$ and $\eta = 1$ at the station where the mixing layers coalesce is $\frac{1}{2}(1{\cdot}19+0{\cdot}135)/(1{\cdot}19-0{\cdot}33) = 0{\cdot}770$ times the initial thickness of the uniform jet.

In general, however, the velocity profile at the slot is not completely uniform because of the effect of boundary layers on the walls of the contraction from the settling chamber to the slot. This may be accounted for by matching the measured velocity profile at the slot to the velocity distribution of the uniform velocity jet at some downstream station, as shown in figure 2. That is, the measured velocity profile at the slot is matched to a velocity profile consisting of a uniform velocity core bounded by two mixing layers of the type investigated by Liepmann & Laufer. When this is done, it is found that the actual edges of the slot correspond to the points $\xi = -0{\cdot}88$ of the matched mixing layers. The distance between the points $\xi = -0{\cdot}88$ and $\xi = 0{\cdot}33$ of the mixing layers is

$$(0{\cdot}33+0{\cdot}88)\,t_m = 1{\cdot}21t_m,$$

where t_m measures the distance between the points $\xi = 0$ and $\xi = 1$. Therefore, if t is the actual slot thickness, the distance between the lines $\xi = 0{\cdot}33$ of the two matched mixing layers in $(t-2\times 1{\cdot}21t_m) = (1-2{\cdot}42c)\,t$, where $c = t_m/t$ gives the ratio of boundary-layer thickness to slot width for the initial velocity profile of the jet. Since the lines $\xi = 0{\cdot}33$ are parallel to the potential core for the initial development of a uniform velocity jet, the measured velocity profile at the slot can be matched to the velocity distribution at a downstream station of the flow of a uniform velocity jet of initial thickness $(1-2{\cdot}42c)\,t$. This matching station is at a distance $12{\cdot}0\,t_m$ from the origin of the uniform velocity jet, since the mixing layers grow at a rate given by $\xi = 12{\cdot}0\;Y/X$. Also, at the station where the mixing layers coalesce, the distance between the points $\xi = 0{\cdot}33$ and $\xi = 1{\cdot}19$ is $\frac{1}{2}(1-2{\cdot}42c)\,t$, so that this station is a distance

$$12{\cdot}0\times\tfrac{1}{2}(1-2{\cdot}42c)t/(1{\cdot}19-0{\cdot}33) = (6{\cdot}97-16{\cdot}9c)\,t$$

downstream of the origin of the uniform velocity jet. Hence the distance between the slot and the point where the mixing layers coalesce is given by

$$X_c = (6{\cdot}97-16{\cdot}9c)\,t-12{\cdot}0t_m,$$

or

$$X_c = (6{\cdot}97-28{\cdot}9c)\,t. \tag{1}$$

Now, since the distance between the points $\eta = 0$ and $\eta = 1$ at the station where the mixing layers coalesce is $0{\cdot}770(1-2{\cdot}42c)\,t = (0{\cdot}770-1{\cdot}87c)\,t$ (since this is $0{\cdot}770$ times the thickness of the uniform velocity jet as shown above), it is seen that the virtual origin of the fully developed jet is at a distance

$$X_0 = (0{\cdot}770-1{\cdot}87c)\,t/E \tag{2}$$

upstream of this station.

It is also necessary to calculate the position of the dividing streamline at the station X_c where the mixing layers coalesce. The dividing streamline is that which springs from the lower edge of the slot, and this is found to correspond to the point $\xi = -0{\cdot}73$ of the mixing layer which is matched to the measured velocity profile at the slot. It should be noted that this does not quite correspond to $\xi = -0{\cdot}88$, the position of the edge of the slot, since a slight adjustment must be made to equate the volume flow in the two matched velocity distributions. Liepmann & Laufer found that $\xi = 0{\cdot}125$ is a streamline for the mixing layer flow, and so, since the volume flow per unit span between the line $\xi = 0{\cdot}125$ and the dividing streamline is constant, at the station X_c the dividing streamline is at $\xi = \xi_{R_0}$ where

$$t_m\int_{-0{\cdot}73}^{0{\cdot}125}\frac{u}{U}\,d\xi = \frac{\frac{1}{2}(1-2{\cdot}42c)\,t}{(1{\cdot}19-0{\cdot}33)}\int_{\xi_{R_0}}^{0{\cdot}125}\frac{u}{U}\,d\xi,$$

or

$$\int_{\xi_{R_0}}^{0{\cdot}125}\frac{u}{U}\,d\xi = \frac{1{\cdot}72c}{(1-2{\cdot}42c)}\,0{\cdot}308 = \frac{0{\cdot}530c}{(1-2{\cdot}42c)}.$$

Writing $u/U = \frac{1}{2}(1+\xi)$ for the region near $\xi = 0{\cdot}125$, it is seen that

$$(1+\xi_{R_0})^2 = 1{\cdot}27 - \frac{2{\cdot}12c}{(1-2{\cdot}42c)},$$

or

$$\xi_{R_0} = \left\{\left(\frac{1{\cdot}27-5{\cdot}18c}{1-2{\cdot}42c}\right)^{\frac{1}{2}} - 1\right\}.$$

This corresponds to

$$\eta_{R_0} = 0{\cdot}804 + 0{\cdot}755\left\{\left(\frac{1{\cdot}27-5{\cdot}18c}{1-2{\cdot}42c}\right)^{\frac{1}{2}} - 1\right\},$$

or

$$\eta_{R_0} = 0{\cdot}049 + 0{\cdot}851\sqrt{\left(\frac{1-4{\cdot}08c}{1-2{\cdot}42c}\right)}. \tag{3}$$

Thus $\eta_{R_0} = 0{\cdot}801$ and $T_0 = \tanh\eta_{R_0} = 0{\cdot}665$ if $c = 0{\cdot}1$. For the experiments previously reported, it was found that the initial velocity profiles correspond to $c = 0{\cdot}1$ for all except the smallest slot widths, in which case $c = 0{\cdot}12$ gives a more accurate profile match.

Although the above procedure may appear to be unduly complicated in view of gross assumptions made elsewhere, it is necessary to calculate the initial growth of the jet to a fair degree of accuracy, particularly for applications of the analysis to other flow configurations in which the slot width t is the only length scale.

Using equation (3) to define the position of the reattaching streamline at station $X = X_0$ (where the co-ordinate X now measures distance along the jet from the virtual origin of the fully developed region), it is seen that the reattaching streamline is at the position $Y = Y_R$ for general X, where

$$\int_{Y_R}^{\infty} u\,dY = \int_{X_0}^{X} E_1 U\,dY + \text{const.},$$

and where the constant is chosen so that Y_R corresponds to η_{R_0} at $X = X_0$. In this equation E_1 is the entrainment parameter for the inner edge of the curved jet. If $T_0 = \tanh\eta_{R_0}$ and $T_1 = \tanh\eta_{R_1}$, where $\eta = \eta_{R_1}$ gives the position of the reattaching streamline at $X = X_1$, the above equation leads to

$$X_0/X_1 = \{(2E_1/E - 1 + T_1)/(2E_1/E - 1 + T_0)\}^2, \tag{4}$$

since the jet momentum per unit span $J = \frac{4}{3}\rho U^2 EX$ is constant along the jet.

The pressure difference Δp across the jet is related to the radius of curvature of the jet centre-line by the equation

$$\Delta p = J/R. \tag{5}$$

At station X_1, the thickness of the jet δ_1, as measured by the distance between the points where $u/U = 0{\cdot}1$, is given by

$$\delta_1/2X_1 = 1{\cdot}825E, \tag{6}$$

and geometrically it is seen that

$$X_1 - X_0 + X_c = R\theta, \tag{7}$$

where θ is the angle between the jet centre-line and the plate at station X_1.

In setting up the momentum equation for the flow parallel to the plate in the vicinity of reattachment, the station X_1, representative of the jet approaching reattachment, is taken to be where the inner edge of the jet, as given by $u/U = 0{\cdot}1$, meets the edge of the reversed flow. The reversed flow profile is assumed to be identical to that part of the jet at X_1 between the reattaching streamline and the point where $u/U = 0{\cdot}1$.

This gives the step height h in terms of θ and R

$$h = R(1-\cos\theta)+\tfrac{1}{2}\delta_1\cos\theta+\tfrac{1}{2}\delta_1(1-\eta_{R_1}/1{\cdot}825). \tag{8}$$

The cavity length l may be taken to be

$$l = R\sin\phi, \tag{9}$$

approximately, where ϕ is the angle between the centre-line of the jet and the plate at the station X_2, where the edge of the jet would first strike the plate if the jet continued to lie on a circle of radius R. Thus

$$X_2-X_0+X_c = R\phi, \tag{10}$$

and

$$h = R(1-\cos\phi)+1{\cdot}825\,EX_2\cos\phi. \tag{11}$$

To allow for the effects of pressure forces near reattachment, the pressure distribution across the jet at X_1 is assumed to be of the form

$$\begin{aligned} p_a-p &= 1/R\int_{-\infty}^{Y}\rho u^2\,dY \\ &= (J/2R)\,(1+\tfrac{3}{2}T-\tfrac{1}{2}T^3), \quad \text{where } T=\tanh\eta. \end{aligned}$$

Integrating this pressure distribution over the parts of the jet below and above the reattaching streamline at X_1 gives the sum of the momentum and pressure forces to be

$$\begin{aligned} &J_2+\int_{Y_{R_1}}^{1\cdot825EX_1}(p-p_a)\,dY \\ &\quad= \tfrac{1}{2}J\{(1-\tfrac{3}{2}T_1+\tfrac{1}{2}T_1^3)-(EX_1/R)\,(3{\cdot}226-\eta_{R_1}-\log\cosh\eta_{R_1}-\tfrac{1}{4}T_1^2)\} \end{aligned}$$

and

$$\begin{aligned} &J_3+\int_{-1\cdot825EX_1}^{Y_{R_1}}(p-p_a)\,dY \\ &\quad= \tfrac{1}{2}J\{(1+\tfrac{3}{2}T_1-\tfrac{1}{2}T_1^3)-(EX_1/R)\,(0{\cdot}424+\eta_{R_1}+\log\cosh\eta_{R_1}+\tfrac{1}{4}T_1^2)\}, \end{aligned}$$

where

$$J+\int_{-1\cdot825EX_1}^{1\cdot825EX_1}(p-p_a)\,dY = J(1-1{\cdot}825\,EX_1/R).$$

If the jet divides in such a way that the sums of the momentum and pressure forces of the parts of the jet below and above the reattaching streamline are conserved (this is equivalent to treating the reattaching jet as a non-dissipative flow), then considering motion parallel to the plate

$$(1-1{\cdot}825EX_1/R)\cos\theta = \tfrac{3}{2}T_1-\tfrac{1}{2}T_1^3-(EX_1/R)\,(\eta_{R_1}+\log\cosh\eta_{R_1}+\tfrac{1}{4}T_1^2-1{\cdot}401). \tag{12}$$

Equations (1)–(12) give $\Delta p/(J/h)$ and l/h as functions of h/t, if E, $2E_1/E$ and c are specified. Thus

$$\frac{\Delta p}{J/h} = 1-\cos\theta+\frac{EX_1}{R}\{1{\cdot}825(1+\cos\theta)-\eta_{R_1}\} = P, \quad \text{say,} \tag{13}$$

$$l/h = \sin\phi/P \tag{14}$$

and

$$h/t = (0{\cdot}770-1{\cdot}87c)\left(\frac{2E_1/E-1+T_0}{2E_1/E-1+T_1}\right)^2\frac{R}{EX_1}P, \tag{15}$$

where

$$EX_1/R = E\theta\Bigg/\left\{1-\left(\frac{2E_1/E-1+T_1}{2E_1/E-1+T_0}\right)^2\left(1-\frac{6{\cdot}97-28{\cdot}9c}{0{\cdot}770-1{\cdot}87c}E\right)\right\}, \tag{16}$$

and

$$T_0 = \tanh\left[0{\cdot}049+0{\cdot}851\{(1-4{\cdot}08c)/(1-2{\cdot}42c)\}^{\frac{1}{2}}\right], \tag{17}$$

where θ and T_1 are related through the momentum balance equation (12), and where ϕ is given by

$$1-\cos\theta+(EX_1/R)\{1{\cdot}825(1+\cos\theta)-\eta_{R_1}\} = 1-\cos\phi + (X_1/R+\phi-\theta)\,1{\cdot}825E\cos\phi. \tag{18}$$

By direct comparison with the detailed velocity and pressure distributions obtained for the case $h/t = 5{\cdot}62$, it is found that the two sides of equation (12) differ by about 2 %, that the predictions of the quantity of the initial reversed and initial downstream flow after reattachment differ from the measured values by about 1 %, and that the assumptions regarding the conservation of momentum and pressure forces which lead to the momentum balance equation (12) are accurate to 5 % of the total jet momentum. This is a considerable improvement on the predictions of the original analysis, and so some confidence is placed in the momentum balance equation (12) and in the definition of the station X_1.

3. Comparison with experiment

Figures 3 and 4 show curves of l/h and $\Delta p/(J/h)$ as functions of h/t given by the modified analysis for curvature ratios $2E_1/E = 0{\cdot}8$ and $1{\cdot}0$ and for entrainment constants $E = 0{\cdot}08$ and $0{\cdot}13$. For all these curves, the initial profile-matching parameter c has the value $0{\cdot}1$.

It is seen that there is a marked improvement in the predictions of the analysis (in figures 3 and 4 the curves given by the previous analysis are also shown). For reasons set out later, it is expected that $2E_1/E = 0{\cdot}83$ and $E = 0{\cdot}13$ should give the values for large h/t, and this is seen to be so. If $E = 0{\cdot}13$ as would appear reasonable, since the total entrainment of the curved jet is expected to remain much the same as that of a free jet, the enhanced entrainment along the outer edge being balanced by a reduced entrainment along the inner edge, then the analysis gives best agreement with the experiments if $2E_1/E$ varies from about $0{\cdot}83$ at $h/t = 5$ to a value of $0{\cdot}9$ at $h/t = 15$, and $0{\cdot}83$ for very large h/t.

It is expected that $2E_1/E$ will vary with h/t since it will be a function of the average value of b/R along the jet. If it is assumed that $1-2E_1/E \propto b/R$, where $b/R = \frac{1}{2}(EX_0/R+EX_1/R)$, then it is possible to calculate the variation of $2E_1/E$ if the value for very large h/t is known. In the following section it is shown that the analysis is consistent with experiment if $2E_1/E = 0{\cdot}83$ for very large h/t, and the curves corresponding to the above calculation are also shown in figures 3 and 4.

A change in the value of the profile-matching parameter c from 0·10 to 0·12 is sufficient to account for the variation in the experimental results obtained at

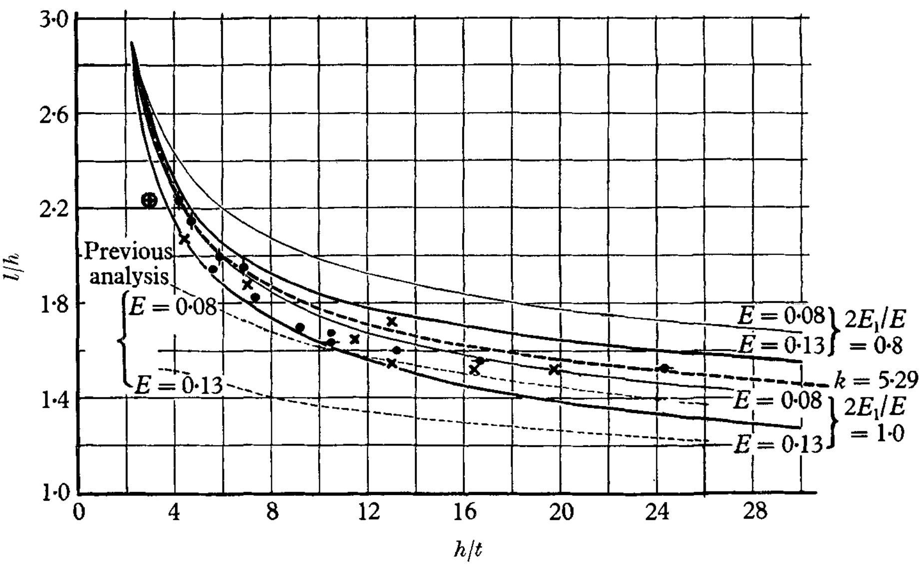

FIGURE 3. Variation of l/h with h/t compared with the analysis. ɸ, ½ in. step; ●, 1 in. step; -●-, 2 in. step; ×, Bourque's measurements; ⊕, Miller & Comings's dual-jet flow.

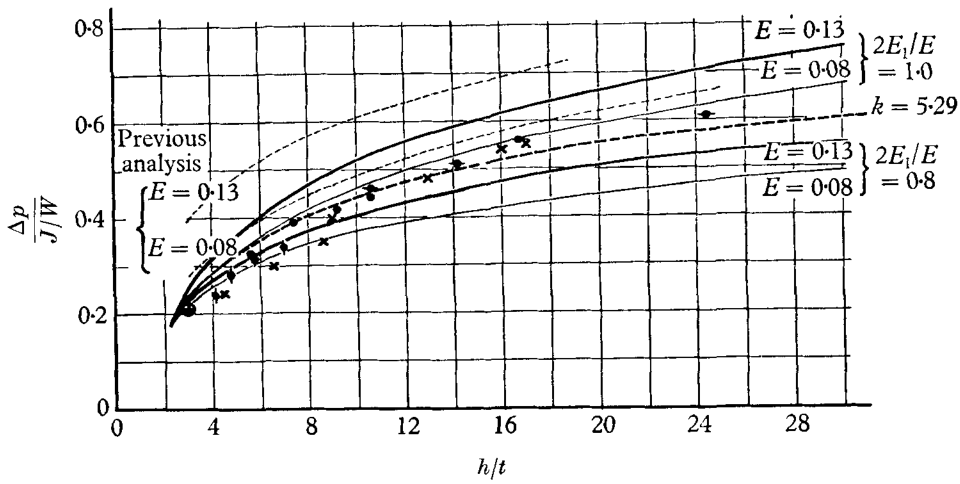

FIGURE 4. Variation of $\Delta p/(J/h)$ with h/t compared with the analysis. ɸ, ½ in. step; ●, 1 in. step; -●-, 2 in. step; ×, Bourque's measurements; ⊕, Miller & Comings.

the same value of h/t but with different step heights. This difference in results is entirely due to the changes in the initial velocity profile.

The dual-jet measurements of Miller & Comings (1960) are also shown in figures 3 and 4.

4. The limit $h/t \to \infty$

From equation (15) it is seen that as $h/t \to \infty$, $T_1 \to 1 - 2E_1/E$. From equations (14) and (13), as $h/t \to \infty$,

$$\left.\begin{aligned} l/h &\to \sin\phi/\{1-\cos\theta + E\theta(1\cdot825(1+\cos\theta) - \eta_{R_1})\}, \\ \text{and}\qquad \Delta p/(J/h) &\to \{1-\cos\theta + E\theta(1\cdot825(1+\cos\theta) - \eta_{R_1})\}, \end{aligned}\right\} \quad (19)$$

where the momentum balance equation (12) gives for the relation between T_1 and θ,

$$(1 - 1\cdot825E\theta)\cos\theta = \tfrac{3}{2}T_1 - \tfrac{1}{2}T_1^3 - E\theta(\eta_{R_1} + \log\cosh\eta_{R_1} + \tfrac{1}{4}T_1^2 - 1\cdot401), \quad (20)$$

and where ϕ is given by

$$\cos\theta - E\theta(1\cdot825(1+\cos\theta) - \eta_{R_1}) = (1 - 1\cdot825E\phi)\cos\phi, \quad (21)$$

from equation (18).

Bourque (1959) publishes measurements of l/h for h/t as large as 50 from which it appears that $l/h \to 1\cdot20$ as $h/t \to \infty$.

Using this information, it is possible to solve equations (19)–(21) to give values of T_1 and θ for each choice of the entrainment constant E. The value of T_1 is very close to 0·170 for E between 0·10 and 0·13 and since $T_1 \to 1 - 2E_1/E$ as $h/t \to \infty$, the entrainment ratio $2E_1/E$ takes the value 0·830 for large h/t. The value of $\Delta p/(J/h)$ is also very nearly constant for E in the range of 0·10 to 0·13 and has a value of 0·80 for very large h/t.

5. A first-order theory for the effects of curvature on mixing processes

The argument put forward by Prandtl showing the mechanism by which curvature increases the turbulent mixing along the outer edge of a curved two-dimensional jet is such as can be readily formulated in terms of simple mixing-length theory.

Consider the transport of a small parcel of fluid from a station at distance $(Y-l)$ from the centre line of the jet to a station Y nearer the outer edge of the jet. This parcel of fluid arrives with an excess velocity of magnitude

$$u(Y-l) - u(Y) = -l(\partial u/\partial Y)$$

over its surroundings. If l is the mixing length then this velocity $-l(\partial u/\partial Y)$ may be taken to be representative of the turbulent velocity u', and correlated with the longitudinal velocity variation u' is a transverse velocity variation v' of the same order of magnitude.

The parcel of fluid is subjected to a centrifugal force of $\{u(Y-l)\}^2/R$ per unit mass as against $\{u(Y)\}^2/R$ of its surroundings. But the pressure gradients and centrifugal forces balance in the mean. Hence the parcel of fluid experiences a total force of $\{u(Y-l)\}^2/R - \{u(Y)\}^2/R = -2lu(\partial u/\partial Y)/R$ tending to move it further outwards. By dimensional arguments, the forces which the parcel of fluid normally encounters will be of the order of u'^2/l per unit mass, since the significant velocity scale for transverse motions will be v' (or u') rather than the absolute velocity u. Since u'^2/l is of order $l(\partial u/\partial Y)^2$, the mixing length has been increased by the factor $\{1-\text{const.}\ 2(u/R)/(\partial U/\partial Y)\}$, and Prandtl's (1925) expression for turbulent shear becomes

$$\tau = \rho l^2 \left|\frac{\partial u}{\partial Y}\right| \left\{\frac{\partial u}{\partial Y} - k\frac{u}{R}\right\}, \tag{22}$$

where k is an empirical constant, and where l is the mixing length for zero curvature.

In obtaining relatively simple analytic expressions for shear-layer mean-velocity distributions, it is much more profitable to use the eddy viscosity rather than the mixing-length. However, the turbulent shear equation derived above may be readily converted to eddy viscosity notation. Thus

$$\tau = \rho\epsilon \left\{\frac{\partial u}{\partial Y} - k\frac{u}{R}\right\}, \tag{23}$$

where ϵ is the eddy viscosity for zero curvature.

Using this expression for the turbulent shear, the equations of motion for curved two-dimensional boundary-layer flow yield similar solutions to the first order in b/R if $b \propto X$, $U \propto V \propto X^{-\frac{1}{2}}$, $(p_0 + \frac{1}{2}\rho\overline{v_0'^2}) \propto X^{-1}$ and b/R is constant, where U, V are the mean velocity components at the jet centre line (the jet centre line is not necessarily a streamline) and p_0, $\overline{v_0'^2}$ refer to the static pressure and mean square fluctuation of v at the jet centre line. It has been assumed that the eddy viscosity for zero curvature is constant across the jet and has the form $\epsilon = \chi b U$, where χ is a constant (Görtler 1942).

The condition that b/R is constant does not, of course, apply to a curved jet across which there is a constant pressure difference. However, first-order solutions to the equations of motion may be obtained for wall jets for which b/R is maintained constant (Sawyer 1962). These solutions, which apply to the outer part of the wall jets, are functions of $(\sigma V/U)$ and b/R, where σ is the spread parameter given by $b = X/\sigma$ and where the transverse length scale is chosen so that $4\sigma\chi = 1$, in line with Görtler's solution for zero curvature.

It might be expected that the velocity profile on one half of a jet with constant curvature would be similar to the solution for constant b/R at least for the outer part of the jet, if the relevant value of b/R is taken to be some average value over the jet up to the station considered. Thus the jet velocity profile should be like that obtained by placing together the two solutions for equal and opposite curvature and, say, average b/R over the jet. Continuity indicates that $\sigma V/U$ should be equal and opposite for the two half jets.

In this case, a relation between the cross-velocity parameter $\sigma V/U$ and the curvature ratio b/R may be found by dimensional arguments applied to the entraining layers. Thus the entrainment or inflow velocity is expected to be proportional to $l(\partial u/\partial Y)$ where l is the mixing length for curved flow, that is, proportional to $\{\partial/\partial\eta\,(u/U) - \frac{1}{2}k(b/R)\,(u/U)\}$ where $\eta = Y/b$, at the centre of the entraining layer. This condition leads to

$$\sigma V/U = (0{\cdot}135k - 1{\cdot}47)\,b/R. \tag{24}$$

The solution for the curved jet gives the entrainment ratio

$$2E_1/E = 1-(0\cdot68k-0\cdot97)(b/R) \tag{25}$$

and the change in half-jet thickness

$$Y_{\frac{1}{2}}/Y_{\frac{1}{2}0} = 1+(1\cdot70k-8\cdot76)(b/R) \tag{26}$$

to the first order in b/R, where $Y_{\frac{1}{2}}$ measures the distance between the points where $u = U$ and $u = \frac{1}{2}U$.

For the case in question, if b/R is taken as the average of the value over the jet and $E = 0\cdot13$, then $b/R = 0\cdot065$ for large h/t, since $\theta \to 57°$ as $h/t \to \infty$. Also since $2E_1/E \to 0\cdot830$ as $h/t \to \infty$, equation (25) is satisfied if $k = 5\cdot29$.

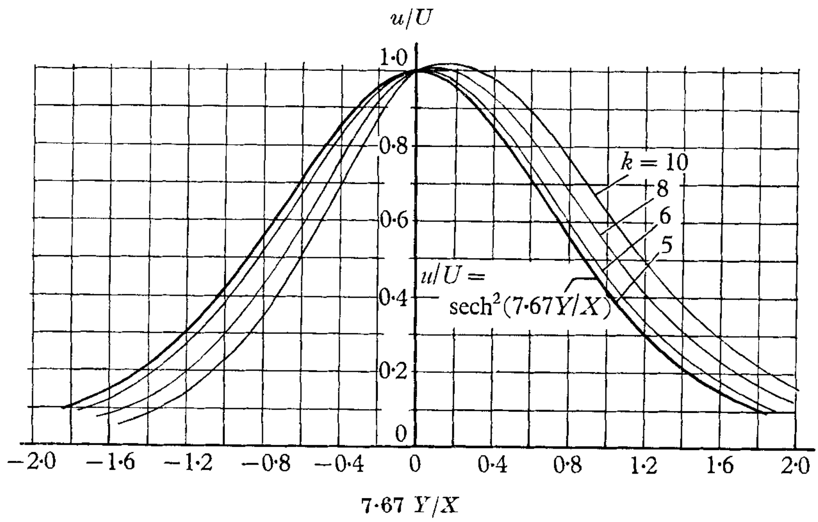

FIGURE 5. Theoretical jet velocity profiles for $b/R = 0\cdot04$.

Using this value for k, equation (26) gives $Y_{\frac{1}{2}}/Y_{\frac{1}{2}0} = 1+0\cdot23\,(b/R)$. This shows that the alteration in half-jet thickness is less than $1\frac{1}{2}\,\%$, which is in agreement with the measured velocity profiles. The rather surprising observation that the velocity profiles are almost identical with those of a free jet is thus shown to be due to the effect of a cross-flow at the locus of maximum profile velocity of the jet just balancing the effect of different entrainment rates at the two sides of the jet.

Equation (24) shows that $V/U = -0\cdot76b/\sigma R$ for $k = 5\cdot29$, so that

$$V/U = -0\cdot007$$

for large h/t, taking $\sigma = 7\cdot67$. Thus the angle between the streamlines and the locus of maximum profile velocity is of the order of $0\cdot4°$. This is in agreement with experiment.

Figure 5 shows jet velocity profiles calculated by the above procedure for the curvature ratio $b/R = 0\cdot04$ and for $k = 5$, 6, 8 and 10. The curve for $k = 5$ is identical with the free-jet velocity profile, and this is found to be so for all practical values of b/R.

6. The flow due to a two-dimensional jet at an angle to a flat plate

The analysis set out above may be adapted to predict overall flow parameters for other situations involving two-dimensional jets, for both steady and quasi-steady mean flows. In what follows, the case of a jet reattaching to an inclined flat plate is considered.

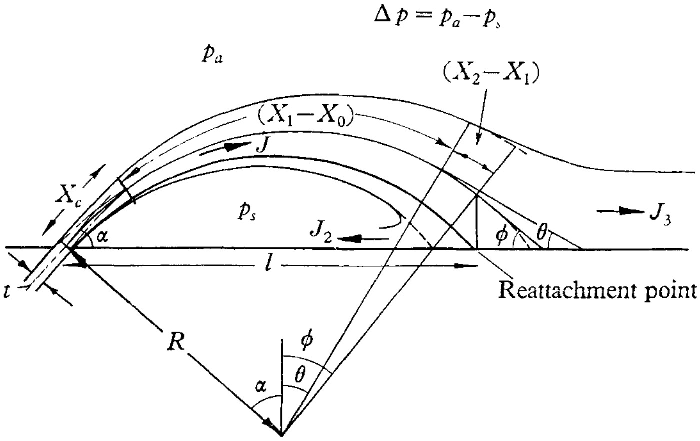

FIGURE 6. Notation used in the analysis of the flow of a jet at an angle to a plate.

The flow pattern and notation used are shown in figure 6. Initially the jet lies approximately in the plane at an angle α to the plate. The proximity of the plate to one side of the jet restricts the inflow of fluid to the jet on that side. This results in a pressure difference across the jet which curves the jet towards the plate, enhancing the restriction of the flow. The curvature of the jet increases until the jet strikes the plate, when a proportion of the volume flow is fed back into the cavity at the reattachment point.

There is a significant difference in the configuration in that there is no longer a reference length equivalent to h of the previous flow pattern. Dimensional considerations indicate that the non-dimensional cavity length l/t and cavity pressure $\Delta p/(J/t)$ are functions of α, the angle between the initial jet direction and the plate, the Reynolds number of the jet $(J\rho t)^{\frac{1}{2}}/\mu$, and the shape of the initial jet velocity profile. If the jet Reynolds number is large, it is expected that both l/t and $\Delta p/(J/t)$ should be independent of $(J\rho t)^{\frac{1}{2}}/\mu$, although both quantities may depend quite critically on the shape of the initial velocity profile.

The previous analysis remains unaltered, except that the cavity length is given by

$$l = R(\sin\alpha + \sin\phi),$$

that equations (7) and (10) are now

$$X_1 - X_0 + X_c = R(\alpha + \theta) \quad \text{and} \quad X_2 - X_0 + X_c = R(\alpha + \phi)$$

and that α and θ are connected by the geometrical relation

$$1 - \cos\alpha = 1 - \cos\theta + (EX_1/R)\{1{\cdot}825(1 + \cos\theta) - \eta_{R_1}\}.$$

Thus the cavity pressure and the cavity length are given by

$$\frac{\Delta p}{J/t} = (\alpha+\theta)\Big/\left\{(6\cdot97-28\cdot9c)+\frac{(0\cdot770-1\cdot87c)}{E}\left[\left(\frac{2E_1/E-1+T_0}{2E_1/E-1+T_1}\right)^2-1\right]\right\}$$

$$= Q, \quad \text{say}, \tag{27}$$

and

$$l/t = (\sin\alpha+\sin\phi)/Q, \tag{28}$$

where

$$EX_1/R = E(\alpha+\theta)\Big/\left\{1-\left(\frac{2E_1/E-1+T_1}{2E_1/E-1+T_0}\right)^2\left(1-\frac{6\cdot97-28\cdot9c}{0\cdot770-1\cdot87c}E\right)\right\},$$

$$T_0 = \tanh\{0\cdot049+0\cdot851\sqrt{(1-4\cdot08c)/(1-2\cdot42c)}\},$$

and

$$1-\cos\alpha = 1-\cos\phi+(X_1/R+\phi-\theta)\,1\cdot825\cos\phi.$$

Values of l/t and $\Delta p/(J/t)$ as functions of α obtained by Bourque (1959) (see also Bourque & Newman 1960) are shown in figures 7 and 8. In figure 8 are presented Bourque's values of $\Delta p/(J/t)$ calculated using the pressure minimum on the plate and also using the pressure on the plate near the jet exit, which he considers to represent more nearly the average pressure in the cavity. In figure 7

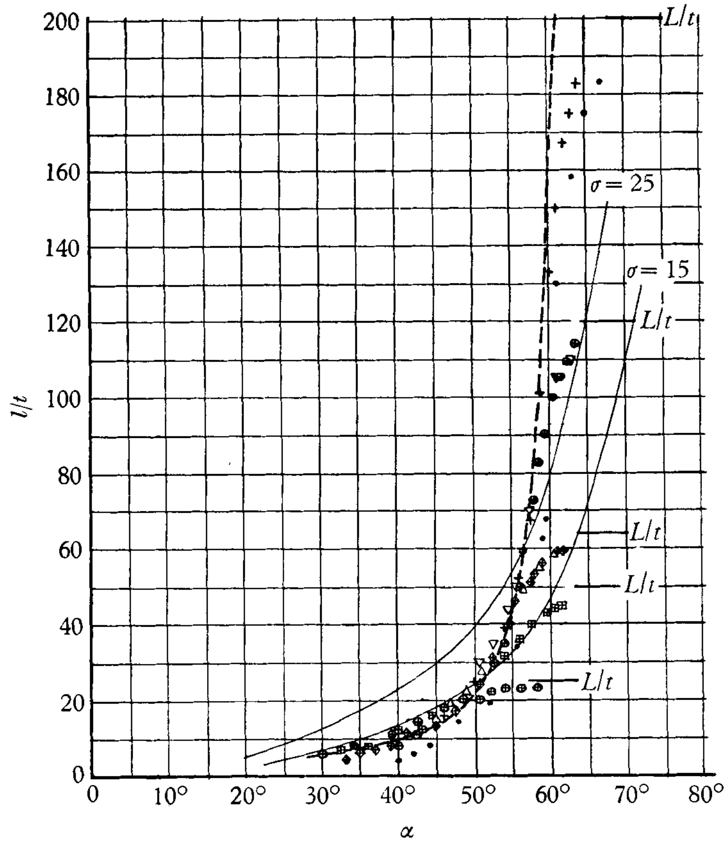

FIGURE 7. Comparison of Bourque's measurements of l/t with the analysis for jet at an angle to a plate. ---, Analysis with $E = 0\cdot130$ and $2E_1/E = 0\cdot638$; ———, Bourque's analysis. The various symbols denoting measured points refer to different Reynolds numbers.

Bourque indicates the points where the length L of the plate used in the experiments acts as a limit to the length of the cavity.

From figure 6 it is seen that $l/t \to \infty$ as $\alpha \to 64^\circ$, and taking $E = 0{\cdot}130$, the above equations indicate that $T_1 \to 0{\cdot}362$, so that $2E_1/E = 0{\cdot}638$ for very large l/t.

The equations have been solved for $E = 0{\cdot}130$, $2E_1/E = 0{\cdot}638$ and $c = 0{\cdot}1$ and the curves for l/t and $\Delta p/(J/t)$ are shown in figures 7 and 8. There is good agreement with the experimental values of l/t, and the theoretical curve for $\Delta p/(J/t)$ lies within the region of uncertainty existing in the experimental results. For comparison, theoretical curves given by Bourque using an analysis similar to that previously given by the author are also included.

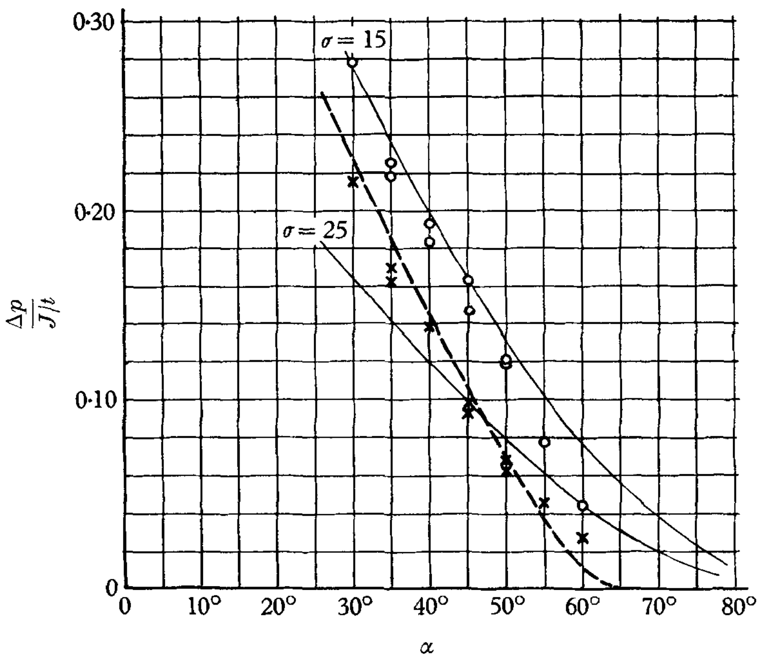

FIGURE 8. Comparison of Bourque's measurements of $\Delta p/(J/t)$ with the analysis for jet at an angle to a plate: ○, corresponding to pressure minimum on plate; ×, corresponding to pressure near jet exit; - - - -, analysis with $E = 0{\cdot}130$ and $2E_1/E = 0{\cdot}638$; ———, Bourque's analysis.

Equation (25) is satisfied for very large l/t if $k = 7{\cdot}87$, since $2E_1/E = 0{\cdot}638$ and $(\alpha+\theta) = 74{\cdot}4^\circ$ in this case, if the relevant value of b/R is taken to be the average over the whole of the jet. For $k = 7{\cdot}87$, the theoretical velocity profiles exhibit some asymmetry, and this is in good agreement with Bourque's measurements. Curves calculated assuming that $2E_1/E = 1-(0{\cdot}68k-0{\cdot}97)\,b/R$, where

$$b/R = \tfrac{1}{2}(EX_0/R+EX_1/R) \quad \text{and} \quad k = 7{\cdot}87,$$

are almost identical with those given by $2E_1/E = 0{\cdot}638$ for all values of α.

The difference in the values of k deduced from the experiments in the above manner may be due to inadequacies of the flow models, or more directly to the fact that the first-order theory is not strictly applicable to jets with constant curvature. Experiments with wall-jets of constant b/R indicate that the higher value for k more nearly represents the effects of curvature on turbulent mixing processes within the framework of simple mixing-length theory. However, it has been shown that the known difference in rates of entrainment at the two edges of two-dimensional jets of constant curvature is not at all incompatible with the observation of nearly symmetrical velocity profiles.

7. Conclusions

The modifications to the analysis given previously result in improved predictions of length and average pressure of the recirculation region associated with the flow of a jet issuing parallel to a flat plate. Best agreement with experiment is obtained when the overall entrainment parameter E has the value 0·130 (that for a plane jet) and the entrainment ratio $2E_1/E$ varies from 0·83 at $h/t = 5$ to 0·9 at $h/t = 15$ and 0·83 for very large h/t. Comparison with detailed flow measurements indicates that the flow model represents the more important features of the flow with a fair degree of accuracy.

The analysis gives good predictions of length and average pressure of the recirculation region for the flow due to a jet emerging at an angle α to a flat plate if $E = 0{\cdot}130$ and $2E_1/E = 0{\cdot}638$ for all angles α for which the jet reattaches to the plate.

A first-order mixing length theory indicates that the effect of different entrainment rates at the two edges of a curved jet does not lead to jet velocity profiles that are substantially different from those of a plane jet, since there is an associated flow across the locus of maximum profile velocity as had been deduced from previous observations.

Curvature is found to have a considerable effect on the entrainment properties of jets, and this is significant in a number of situations, for example, the growth of curved wall jets, the curved-jet recirculation flows associated with air cushion vehicles, and the velocity fields induced by curved jets.

The author wishes to express his appreciation of many stimulating discussions with Dr M. R. Head, who first brought to the author's notice the importance of curvature effects on entrainment properties, and also to thank the Department of Scientific and Industrial Research for their financial assistance.

Appendix. List of symbols

b	thickness of jet (distance between points $\eta = 0, 1$)
c	initial jet velocity profile parameter, $c = t_m/t$
E	overall jet entrainment parameter
E_1, E_2	entrainment parameters for inner and outer edges of curved jet
h	step height
J	jet momentum per unit span
J_2, J_3	momentum per unit span of initial reversed and downstream flow
k	empirical constant
l	length of recirculation region, also mixing length (§ 5)

p	static pressure
Δp	pressure difference across curved jet
R	radius of curvature of jet centre line
t	slot thickness
t_m	thickness of mixing layer portion of initial jet profile
T	$T = \tanh \eta_R$
T_0, T_1	values of T at stations X_0, X_1
u	velocity parallel to jet centre line
U	maximum jet profile velocity
v	velocity perpendicular to jet centre line
V	value of v at $\eta = 0$
X	co-ordinate measuring distance parallel to jet centre line
X_0	value of X corresponding to position where initial mixing layers coalesce
X_1	value of X corresponding to state of jet at reattachment
X_2	value of X giving reattachment position
Y	co-ordinate measuring distance perpendicular to jet centre line
Y_R	value of Y corresponding to position or reattaching streamline
$Y_{\frac{1}{2}}$	half-jet thickness, distance between points where $u/U = 1, \frac{1}{2}$
$Y_{\frac{1}{2}0}$	value of $Y_{\frac{1}{2}}$ for zero curvature
α	angle between initial jet direction and plate
δ_1	jet thickness at station X_1
ϵ	eddy viscosity
η	non-dimensional Y co-ordinate, $\eta = Y/EX$
η_R	value of η corresponding to reattaching streamline
η_{R_0}, η_{R_1}	value of η_R at stations X_0, X_1
θ	angle between jet centre-line and plate at station X_1
μ	fluid viscosity
ξ	non-dimensional mixing-layer co-ordinate, $\xi = 12{\cdot}0\,Y/X$
ρ	fluid density
σ	empirical constant giving rate of spread of jet
τ	turbulent shear
ϕ	angle between jet centre line and plate at station X_2
χ	empirical constant giving eddy viscosity, $\epsilon = \chi b U$.

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